ometry are desirable as they save time. This presentation of the matter is offered mainly on account of its theoretic interest, and because it connects this elementary process in trigonometry with the extensive subject of group theory. In fact, the operations denoted by c and s generate a group of order 8 known as the octic group or the group of the square. From this fact it results directly that the operations c and s could be replaced by any pair of generators of this octic group. In particular, we could replace Table III by one giving the functions of A in terms of functions of $-A$, without changing tables I and II.

The point of view developed above has other advantages as it directs attention to the fact that we replace Table I by one covering a very much smaller interval provided we replaced the other two tables by suitable tables. In fact, Table I could be diminished indefinitely, but such a reduction would have no practical bearing. To indicate changes along this line, which have only theoretical interest, we may observe that Table I could be replaced by one covering only the interval from 0° to $22\frac{1}{2}^\circ$, by assuming Table II and replacing Table III by the following:

$$\sin(45^\circ - x) = \tfrac{1}{2}\sqrt{2}(\cos x - \sin x), \quad \cot(45^\circ - x) = \frac{\cos x + \sin x}{\cos x - \sin x},$$

$$\cos(45^\circ - x) = \tfrac{1}{2}\sqrt{2}(\cos x + \sin x), \quad \sec(45^\circ - x) = \frac{\sqrt{2}}{\cos x + \sin x},$$

$$\tan(45^\circ - x) = \frac{\cos x - \sin x}{\cos x + \sin x}, \quad \csc(45^\circ - x) = \frac{\sqrt{2}}{\cos x - \sin x}.$$

In fact, the operations of getting the complement and of subtracting from 45° generate a group of order 16, as may readily be verified,* and by these operations an angle is, in general, transformed into one and into only one angle in each of the sixteen equal divisions of the circumangle, starting from 0°. Hence we can find the trigonometric functions of any angle if we know the functions of all the angles in one of those intervals of $22\frac{1}{2}^\circ$ and the operations expressed in the table ending the preceding paragraph, together with those of Table II. There is evidently no limit to similar developments along these lines.

One of the many general methods which includes the special cases given above and illustrates how the interval of the angles whose trigonometric functions are supposed to be known may be reduced indefinitely, is as follows: The group of movements of the regular polygon of n sides is the dihedral group of order $2n$. This group transforms *transitively* each of the

* *Annals of Mathematics*, Vol. 8 (1907), p. 97.

$2n$ half sides of the regular polygon; that is, it may be represented as a regular substitution group on letters so selected that each letter corresponds to one of these half sides. Hence *the group of movements of the regular polygon contains one and only one operator which transforms a particular half side of the polygon into a given half side.* As this group of movements is dihedral it may be generated by two of its operators of order two.

It is known that subtraction may be regarded geometrically as a reflection of the point representing the subtrahend on the point midway between the minuend and the origin.* If we inscribe in a circle a regular polygon of $2m$ sides, $m>1$, the group of movements of this polygon is clearly generated by a rotation through π on the line of symmetry bisecting the first side and a rotation through π on the y-axis. These rotations are equivalent, respectively, to subtraction from π/m and subtraction from π. If we make $m=2$ we have the group, considered above, formed by the operations of getting the complement and the supplement.

In the second example considered above, $m=4$, and, instead of subtracting from π we subtracted from $\frac{1}{2}\pi$, which is evidently always permissible when m is even and greater than 2. As the table of the trigonometric functions may be confined to the interval $\pi/2m$ provided we can deduce the functions of $\pi-A$, and of $\pi/m-A$ from those of A, this indicates how the necessary Table I may be reduced theoretically to an interval which is less than any given interval. Practically, this method should probably not be employed beyond $m=2$.

* *Annals of Mathematics*, Vol. 6 (1905), p. 41.

A SET OF INDEPENDENT ASSUMPTIONS FOR PROJECTIVE GEOMETRY.*

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INTRODUCTION.

The purpose of the present paper is sufficiently indicated in the title. Incidentally an elementary statement is made of well-understood principles in the foundations of mathematics, and the actual working of these principles is exhibited in treating the set of assumptions for projective geometry. Other sets of assumptions for projective geometry have been given by Veblen and Young† and by the present writer.‡ It is believed that the set of assumptions given in the present paper will lead to a development of Pro-

* The substance of this paper was presented to the American Mathematical Society in February, 1911.

† Veblen and Young, *Projective Geometry*, Vol. I.

‡ Lennes, "Duality in Projective Geometry," *Annals of Mathematics*, Vol. 11 (1911).

jective Geometry which shall be simpler and at the same time form a more natural sequel to Euclidean Geometry than the developments resulting from the other sets just mentioned.

It is usual to begin any logical discussion such as the present one with a set of definitions of the technical terms to be used. It is obvious, however, that any definition of such a term can only be made in terms of other technical terms which are equally in need of definition. The definitions of these terms will introduce still other terms which in turn require definition, and so on. Hence the process of definition never terminates. One alternative would be defining in a circle which is not permitted in a logical science. The other alternative is the one in use among mathematicians of the present day, namely, starting with a set of undefined terms* which are usually specified at the outset. In this paper the undefined terms are "point," "line," "plane," and an undefined relation "on."

When certain terms have been designated as undefined other terms may be defined in terms of these. Thus, while point, line, and plane are undefined, a *flat pencil of lines* is defined as "the set of all lines in a plane passing through one point of that plane." The objection may now be urged that this gives a definition of flat pencil in terms of undefined words and hence is no definition at all. The answer is that it is the only kind of definition possible.

However the undefined terms are given a sort of indirect definition by the fundamental assumptions in which they occur.† Indeed the undefined terms have no logical content whatever except as it is given them by the assumptions in which they occur.

Clearly assumptions‡ are necessary in any purely deductive system, for a proof in such a system consists simply in showing that according to certain admitted rules of logic a given proposition is a necessary consequence of certain other propositions. Hence it is as impossible to prove every proposition as it is to define every term.

The set of assumptions for a mathematical science must fulfill certain well established requirements. That is, they must be sufficient, independent and consistent. The most important of these is the first, namely, that all the

* It should be clearly understood that there is no such thing as an *undefinable* term. The fact is simply that not *all* terms in any particular logical science can be defined. However, two different writers developing the same subject may use entirely different sets of undefined terms.

† The classical name for what in this paper is called assumption was axiom, which was defined as a self-evident truth. It is obvious, however, that no proposition dealing with abstract, undefined symbols can possibly be self-evident. To avoid the connotation of the old term axiom many recent writers have used other terms such as postulate or assumption. The old Greek distinction between postulate and axiom has entirely disappeared from mathematics (except in some elementary texts whose authors are evidently not acquainted in a vital sense with the modern point of view). This distinction seems to have its basis in a misunderstanding of the nature of unproved propositions in mathematics.

‡ The French term for assumption, which translated reads "unproved proposition," seems the most fortunate of all the names that have been used. An axiom, assumption, postulate, or whatever we call it, is distinguished from the other propositions of the system in which they occur solely by the fact they are accepted arbitrarily and without proof. As a matter of fact there are modern sets of "assumptions" which contain propositions much less obvious to the intuition than are some of the theorems derived from them.

remaining propositions of the science must follow from the assumptions by purely *formal* deduction. This means that at no stage in the argumentation is the intuition or any consideration whatever aside from the propositions used in making the proof permitted to validate a statement or an argument.

The assumptions must thus contain by implication every theorem in the mathematical science in which they occur. This requirement is by no means satisfied by the usual axioms of Euclidean geometry. Thus it is not possible to prove by means of these axioms that the diagonals of a parallelogram meet in a point.

The second requirement, namely, that the assumptions shall be independent, means that no one of them shall be a logical consequence of all the others. The set given in this paper are proved independent in § 4. A proof that one proposition cannot be proved from a given set of propositions is of comparatively recent origin. The well known proof of Lobachevsky (1829) that the assumption of parallels is not a logical consequence of the remaining assumptions of Euclidean geometry is apparently the first one of this kind. Later completely independent systems of assumptions have been given by Hilbert, Veblen, Huntington, and others. The method of proving the independence of a given set of assumptions is discussed in § 4.

A set of assumptions is consistent if no two contradictory propositions can be deduced from them. The method of proving a set of assumptions consistent is discussed in § 3.

A set of assumptions may or may not possess the further quality of being categorical. This property of a set of assumptions is somewhat more difficult to understand than those just described. The aseumptions of the present paper are not categorical. It is shown in § 3 that all these assumptions are satisfied by certain finite sets of points, lines and planes. Obviously they are also satisfied by the ordinary projective geometry. That is, these assumptions are satisfied by two essentially different systems. If, on the other hand, a set of assumptions is such shat any two systems which satisfy all its assumptions are essentially alike, the set is said to be categorical.*

In the present paper it will turn out that the undefined terms point, line, and plane are used in the sense ascribed to them in ordinary projective geometry. A point is "on" a line or plane if it lies in the line or plane in the ordinary sense. In this case the line and plane are also said to be "on" the point. A line is "on" a plane if it lies in the plane, and in this case the plane is also said to be "on" the line. The statements just given are by no means definitions in the logical sense, but merely informal descriptive statements to aid the intuition in giving concrete form to the abstract symbols and propositions that follow. It should be noted that such concrete form is no part of the logical proceedure but simply an aid to the memory in retaining the various steps.

* For a more precise statement of this matter see E. V. Huntington, *Annals of Mathematics*, Vol. 6 (1906), pp. 1-43.

One of the most far-reaching theorems of Projective Geometry is the so-called Principle of Duality, namely, *Any valid proposition stated in terms of "point," "line," and "plane" and the relation "on" is valid if point and plane are interchanged.*

In the older treatments of projective geometry this general proposition was not proved, the writers contenting themselves with writing the dual theorems and their proofs side by side, thus exhibiting the principle as each theorem arose. In the treatment of Veblen and Young noted above the theorem is proved in its entirety but only after a considerably difficult and complicated argumentation. The undefined terms of their system are "point," and "line," and the relation "on." "Plane" is then defined as a class of points by means of certain collinearities. Since point is an undefined term, while plane is defined, a completely dual treatment from the outset is impossible.

In the present paper the fundamental propositions form a dual set. That is, they are not changed if the words point and plane are interchanged throughout. Hence it follows without further argument that the set of all theorems provable from these them must be dual. That is, for every such theorem containing point and plane there is another exactly like it except that the words point and plane have changed places.

§ 1. Fundamental Propositions.

I. *The relation "on" is a reciprocal relation.*

This proposition may be stated more in detail as follows:

1. "The point A is on the line l" is equivalent to "the line l is on the point A."

2. "The point A is on the plane α" is equivalent to "the plane α is on the point A."

3. "The line l is on the plane α" is equivalent to "the plane α is on the line l."

Each of these propositions may be split into two. Thus I may be written:

(a) "The point A is on the line l" implies "the line l is on the point A."

(b) "The line l is on the point A" implies "the point A is on the line l."

The three propositions (1), (2), (3) will be referred to as I_1, I_2, I_3.

II. *If the point A and the plane α are each on a line l then A is on α.*

III_1. *Any two points are on at least one line.*

III_2. *Any two planes are on at least one line.*

IV_1. *Two points are on not more than one line.*

IV_2. *Two planes are on not more than one line.*

Definitions. Points which are on the same line are *collinear*. Otherwise they are *noncollinear*. Planes which are on the same line are *collinear*, and otherwise they are *noncollinear*.

V_1. *Three points are on at least one plane.*

V_2. *Three planes are on at least one point.*

VI_1. *Three noncollinear points are on not more than one plane.*

VI_2. *Three noncollinear planes are on not more than one point.*

VII. *The class of elements known as lines consists of at least one element.*

$VIII_1$. *On every line there are at least three points.*

$VIII_2$. *On every line there are at least three planes.*

IX_1. *On every plane there are at least three noncollinear points.*

IX_2. *On every point there are at least three noncollinear planes.*

Propositions I, ..., IX_2 are obviously dual with respect to point and plane.

§ 2. Assumptions and Theorems.

$I_{1,2,3}$; II, III_2, IV_1, $V_{1,2}$; VI_1, VII; $VIII_{1,2}$; IX_1 form a set of independent assumptions and all theorems of this section are based on them. III_1; IV_2; VI_2; IX_2 are theorems. For proofs of the independence of the postulates see § 4. III_1, IV_2, VI_2 and IX_2 are proved in this section.

IV_2, *Two planes are on not more than one line.*

Proof. Given the distinct planes α and β which we suppose to be on each of the two lines l_1 and l_2. By $VIII_1$ there are three points on each line, and hence by IV_1 there are among these points a set of three noncollinear points A, B, C. By II each of the points A, B, C is on α and also on β. But by VI_1 A, B, C are on only one plane.

1_1. Theorem. *If a point is on each of two planes then the point is on the line which is on the two planes.*

Proof. Two planes α and β are on one and only one line l (III_2, IV_2); on this line are at least two points A and B. If now there is a point C on both α and β but not on l then A, B, C are noncollinear and hence by VI_1 α and β are the same plane.

VI_2. *Three noncollinear planes are on not more than one point.*

Proof. If there are two points A and B on each of the planes α, β, γ and if α, β are on the line l_1 and β and γ are on the line l_2, then A and B are on l_1 and also on l_2, and hence l_1 and l_2 are identical, and hence α, β, γ are collinear.

2_1. *Not all points on a plane.*

Proof. Let α be any plane. By VII, $VIII_2$ at least a plane β distinct from α. By IX_1, II not all points on β are on α. Hence not all points on α.

III_1. *Any two points are on a line.*

Proof. Let A and B be any two points. By VII a line l exists. If A and B are on l then the theorem is verified. If A and B are not on l there is a point C on l (not collinear with A and B). ($VIII_1$, IV_1). Let D be a point not on the plane α of A, B, C. Then A, B, D are on a plane β. α and β are on one and only one line l_1 (III_2, IV_2). But A and B are on both α and β. Hence l_1 is on A and B (1).

IX_2. *On every point there are at least three noncollinear planes.*

PROOF. Let A be the given point. By VII and $VIII_1$ there exists at least one other point B, and by III_1 there is a line l on A and B. By $VIII_2$ there are three distinct planes α, β, γ on l, and on each plane at least one point not on l. Let C, D, E be such points, one on each of the three planes. Then A, C, D are on a plane Δ. This plane Δ is not on l for in that case it would be on both A and B (II), and hence would be identical with both α and β. But α and β are both on l, whence by (II) any point on both α and β is on l. Therefore Δ is not collinear with α and β. That is, α, β, Δ are three noncollinear planes on A.

Propositions I, ..., IX_2 are now completely established (either by assumption or by proof) and hence we state without further argument the space dual of (1_1).

1_2. THEOREM. *If a plane is on each of two points then the plane is on the line which is on these two points.*

3_1. THEOREM. *If two lines are on a plane they are on a point.*

PROOF. Denote the given plane by α and the given lines on it by l_1 and l_2. By $VIII_2$, IX_1 there is a point C not on α. Let A and B be two points on l_1 and DE two points on l_2. Then ABC are noncollinear as are also CDE (2_1). Denote the planes on these points by β and γ, respectively. By V_2 there is a point F on the planes α, β, γ. Thus F is on l_1 and l_2 (1_2).

3_2. THEOREM. *If two lines are on a point they are on a plane.*

PROOF. This is the dual of (3_1).

Of the usual propositions in projective geometry on the incidence of points, lines, and planes those that now remain are obvious and corollaries of the preceding propositions of this section.

Inasmuch as only a finite number of elements are required to satisfy either set of assumptions (see § 3) some theorems may be trivial, since there may not be a sufficient number of elements present to satisfy the hypothesis. Thus if only the elements required by the assumptions are permitted the Theorem of Desargues is trivial since only seven points are on any one plane.

§ 3. CONSISTENCY OF THE ASSUMPTIONS.

We now inquire whether the given set of assumptions is consistent, independent, and categorical.

That a set of assumptions is consistent is proved by exhibiting a concrete system in which the assumptions are satisfied and which for some reason is regarded as self-consistent. The propositions of § 1 are all satisfied by a system given below in which points are the capital letters, lines the columns each containing three letters, and each set of seven lines forming a rectangle, is a plane.*

* Finite geometries of which the one given here is a type have been studied by a number of writers. Tactical memoranda by E. H. Moore, *American Journal of Mathematics*, Vol. 18 (1896), pp. 264-303, contains a rich collection

(A)

	A B C D E F G		*A B H D I J K*		*A B M D L N O*
(1)	*B C D E F G A*	(2)	*B H D I J K A*	(3)	*B M D L N O A*
	D E F G A B C		*D I J K A B H*		*D L N O A B M*
	B C K E L M J		*B C N E H I O*		*C D I F K L O*
(4)	*C K E L M J B*	(5)	*C N E H I O B*	(6)	*D I F K L O C*
	E L M J B C K		*E H I O B C N*		*F K L O C D I*
	C D H F J M N		*D E O G H J L*		*D E M G K I N*
(7)	*D H F J M N C*	(8)	*E O G H J L D*	(9)	*E M G K I N D*
	F J M N C D H		*G H J L D E O*		*G K I N D E M*
	E F L A I N J		*E F K A O H M*		*F G J B O K N*
(10)	*F L A I N J E*	(11)	*F K A O H M E*	(12)	*G J B O K N F*
	A I N J E F L		*A O H M E F K*		*B O K N F G J*
	F G H B L I M		*G A I C J O M*		*G A K C H L N*
(13)	*G H B L I M E*	(14)	*A I C J O M C*	(15)	*A K C H L N G*
	B L I M F G H		*C J O M G A I*		*C H L N G A K*

In this system there are three planes and also three points on every line. Thus the planes (1), (2), (3) are on the line $\begin{matrix} A \\ B \\ D \end{matrix}$ as are also the points *A, B, D.* There are seven points on each plane and seven planes on every point. That all the assumptions of § 1 are satisfied by this system is easily verified.

§ 4. Independence of the Assumptions.

We now prove that $I_{1,2,3}$; II; III_2, IV_1; $V_{1,2}$; VI_1; VII, $VIII_{1,2}$; IX_1 of § 1 form a set of independent postulates. $I_{1,2,3}$; II; III_2, IV_1; $V_{1,2}$; VI_1 are to be understood as not affirming the *existence* of points, lines and planes. That is, these assumptions would all be satisfied by an entirely vacuous system. For convenience of reference these assumptions are here restated in more explicit form.

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of such configurations though they are not regarded as finite geometries. Moore first used the matricular notation in this connection. Cf. Veblen and Young, *Projective Geometry*, Vol. 1. In this notation the configuration here given is

$$\begin{matrix} 15 & 3 & 7 \\ 3 & 35 & 3 \\ 7 & 3 & 15 \end{matrix}$$

This particular configuration is studied in detail by Moore in a paper on the General Equation of the Seventh and Eighth Degree, *Mathematische Annalen*, Vol. 51 (1899), pp. 417-444. It is also studied by George M. Conwell, *Annals of Mathematics*, Vol. 11 (1910), second series, pp. 60-76. Veblen and Bussey, *Transactions of the American Mathematical Society*, Vol. 7 (1906). pp. 241-259, treated these configurations explicitely as finite geometries and discovered the set of all such geometries.

I_1. *"A point A is on a line l" is equivalent to "the line l is on the point A."*

I_2. *"A point A is on a plane α" is equivalent to "the plane α is on the point A."*

I_3. *"A line l is on a plane α" is equivalent to "the plane α is on the line l."*

II. *If a point A and a plane α are both on a line l then A is on α.*

III_2. *If planes and lines exist then any two planes are on at least one line.*

IV_1. *If points and lines exist then any two points are on not more than one line.*

V_1. *If points and planes exist then any three points are on at least one plane.*

V_2. *If planes and points exist then any three planes are on at least one point.*

VI_1. *If points and planes exist then three noncollinear points are on not more than one plane.*

VII. *The class whose elements are lines contains at least one element.*

$VIII_1$. *On every line there are at least three points.*

$VIII_2$. *On every line there are at least three planes.*

IX_1. *On every plane there are at least three noncollinear points.*

INDEPENDENCE OF I_1. To prove the independence of I_1 modify the concrete system (A) of § 3 so that the line $\begin{matrix} A \\ B \\ D \end{matrix}$ shall not be *on* the point A while A is *on* this line. In this system every arsumption except I_1 is verified.

I_2. To prove the independence of I_2 modify the system (A) so that plane (1) shall be *on* the points on which (2) now is, and (2) shall be *on* the points on which (1) now is. The points (1) and (2) remain *on* these planes.

I_3. Modify the concrete system (A) so that the line $\begin{matrix} A \\ B \\ D \end{matrix}$ is not on the plane (1) but plane (1) is on the line.

II. Modify the system (A) so that the points which are now on plane (1) shall be on plane (2), and those now on plane (2) shall be on plane (1). Likewise plane (1) shall be on the points now on plane (2), and plane (2) on the points now on plane (1). Points and lines, and planes and lines are related as in system (A). Thus the line $\begin{matrix} E \\ F \\ A \end{matrix}$ remains *on* the plane (1) and the points E, F, A are on this line while they are also *on* the plane (3).

III_2. In system (A) omit the line $\begin{matrix} A \\ B \\ D \end{matrix}$ but not the points A, B, D.

IV_1. In system (A) add the point G to any line on which it now is not.

V_1. To the system (A) add a point Q which shall be on no line and on no plane.

V_2. To prove the independence of this assumption considsr an ordinary projective geometry with one point removed. Then any three planes which pass through this point are *on* no common point.

VI_1. In the system (A) let the point C be *on* plane (2) (as well as on plane (1)) but not on any line in plane (2).

VII. An entirely vacuous system containing no points, lines, or planes.

$VIII_1$. In the system (A) omit all points and planes.

$VIII_2$. In the system (A) omit all planes.

IX_1. Consider a system consisting of planes (1), (2), (3) of (A) modified as follows: $\begin{matrix} A \\ B \\ D \end{matrix}$ is a line on each plane and A, B, D are points. The remaining points and lines now in these planes shall be some other distinct entities different from either points or lines. Then we have a system consisting of one line with three points on it, and also three planes. Clearly the points are distinct among themselves, as are also the planes.

Hence we have showed that the set of assumptions of this paper are independent.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

GEOMETRY.

383. Proposed by S. A. COREY, Hiteman, Iowa.

Let $ABCDE$ be a pentagon, plane, or gauche, with sides AB, BC, CD, and DE, of lengths, w, x, y, and z, respectively. Construct four other pentagons, $AB_1C_1D_1E_1$, $AB_{11}C_{11}D_{11}E_{11}$, $AB_{111}C_{111}D_{111}E_{111}$, and $AB_{iv}C_{iv}D_{iv}E_{iv}$, having a common vertex at A, and with four consecutive sides in each parallel to the corresponding consecutive sides, AB, BC, CD, and DE, in $ABCDE$. Further, let the lengths of the sides AB_1, B_1C_1, C_1D_1, D_1E_1, AB_{11}, $B_{11}C_{11}$, $C_{11}D_{11}$, $D_{11}E_{11}$, AB_{111}, $B_{111}C_{111}$, $C_{111}D_{111}$, $D_{111}E_{111}$, AB_{iv}, $B_{iv}C_{iv}$, $C_{iv}D_{iv}$, $D_{iv}E_{iv}$, be $-wW$, xX, yY, zZ, wX, xW, yZ, $-zY$, wY, yW, zX, $-xZ$, wZ, zW, xY, and $-yX$, respectively; the minus sign indicating the reversal of direction of the corresponding side. Prove that $(W^2+X^2+Y^2+Z^2)(w^2+x^2+y^2+z^2)=AE_1^2+AE_{11}^2+AE_{111}^2+AE_{iv}^2$.

Solution by the PROPOSER.

Let **w**, **x**, **y**, and **z** represent the vector sides, AB, BC, CD, and DE, respectively, of the pentagon $ABCDE$, the tensors of corresponding vectors being w, x, y, and z. Then in the other four pentagons will $-W\mathbf{w}$, $X\mathbf{x}$, $Y\mathbf{y}$, $Z\mathbf{z}$, $X\mathbf{w}$, $W\mathbf{x}$, $Z\mathbf{y}$, $-Y\mathbf{z}$, $Y\mathbf{w}$, $W\mathbf{y}$, $X\mathbf{z}$, $-Z\mathbf{x}$, $Z\mathbf{w}$, $W\mathbf{z}$, $Y\mathbf{x}$, and $-Y\mathbf{y}$, represent the vector sides AB_1, B_1C_1, C_1D_1, D_1E_1, AB_{11}, $B_{11}C_{11}$, $C_{11}D_{11}$, $D_{11}E_{11}$, AB_{111}, $B_{111}C_{111}$, $C_{111}D_{111}$, $D_{111}E_{111}$, AB_{iv}, $B_{iv}C_{iv}$, $C_{iv}D_{iv}$, and $D_{iv}E_{iv}$, respectively. The remaining or fifth side of each of the four pentagons may be represented as the sum of the other four vector sides, whence

$$AE_1=-W\mathbf{w}+X\mathbf{x}+Y\mathbf{y}+Z\mathbf{z}, \quad AE_{11}=X\mathbf{w}+W\mathbf{x}+Z\mathbf{y}-Y\mathbf{z},$$
$$AE_{111}=Y\mathbf{w}+W\mathbf{y}+X\mathbf{z}-Z\mathbf{x}, \quad AE_{iv}=Z\mathbf{w}+W\mathbf{z}+Y\mathbf{x}-X\mathbf{y}.$$

Substituting the squares of these four vector sums for the squares of the fifth sides which constitute the second member of the given equation, expanding and adding, their sum is found to be identical with the first member, due regard being had to the change of signs. Q. E. D.

The foregoing non-geometric solution is given to show the advantage sometimes resulting from the introduction of vectors in the handling of certain classes of geometric problems. In this case the remarkable simplicity and directness of the solution is due to the fact that the equation given in the problem is a geometric interpretation of the algebraic identity,

$$(W^2+X^2+Y^2+Z^2)(w^2+x^2+y^2+z^2)=(-wW+xX+yY+zZ)^2$$
$$+(wX+xW+yZ-zX)^2+(wY+yW+zX-xZ)^2+(wZ+zW+xY-yX)^2.$$

We may observe that every algebraic identity, in which each term is of the second degree in so far as certain letters are concerned, may be given a geometric interpretation if each of such letters be used to represent a certain vector, and if the scalar but not the vector portion of the product be employed in the interpretation. That this is true follows from the fact that the non-commutative character of vector multiplication does not alter or affect the scalar portion of the product, if each term of such product contains either the product of two separate vectors or the square of some one vector; *i. e.*, if no term in the expanded form is of a degree higher or lower than the second in the letters used to designate vectors.

A solution similar to the foregoing may be employed in problem 377.

384. Proposed by S. LEFSEHETZ, Clark University.

Let ABC be a triangle, O a circle tangent to its three sides, T a variable tangent of O, which cuts the sides BC, CA, AB in a, b, c. Oa', Ob', Oc' the perpendiculars in O to Oa, Ob, Oc, cutting, respectively, T in points a', b', c'. Prove that Aa', Bb', Cc' meet in a point t, and find the locus of t when T varies. Purely geometrical proofs wanted.

No solution of this problem has been received.

385. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

Given a triangle ABC, find the radius of a circle touching two of its sides and a line parallel to the third, at a distance $d=u+2r$.

Solution by A. H. HOLMES, Brunswick, Maine.

Let a, b, and c be the sides of the given triangle, c the base. Then h=altitude of the triangle$=\frac{\sqrt{[4a^2c^2-(a^2-b^2+c^2)]}}{2c}$, and R=radius of the inscribed circle $=\frac{\sqrt{[4a^2c^2-(a^2-b^2+c^2)]}}{2(a+b+c)}$.

Put r=radius of circle touching a and b and a line parallel to c at a distance from c, $2r+u$. Then $h:R=h-(2r+u):r$.

$$\therefore r=\frac{(h-u)R}{h+2R}.$$

Putting for h and R their values in terms of a, b, and c, we have,

$$r=\frac{\sqrt{[4a^2c^2-(a^2-b^2+c^2)^2]}-u}{a+b+3c}.$$

CALCULUS.

306. Proposed by FRANCIS RUST, C. E., Pittsburg, Pa.

Express in elliptic integrals: $A_\theta=\int_0^\theta \frac{dx}{\sqrt{(1-x^4)}}$; $0<\theta<1$.

I Solution by WALTER D. LAMBERT, A. M., University of Pennsylvania, Philadelphia, Pa.

Correcting the inequality to read $0<\theta<1$, we find (by using the substitution $x=\cos\phi$, and by calling $a=\cos^{-1}\theta$) that

$$A_\theta=\int_0^\theta \frac{dx}{\sqrt{[(1-x^2)(1+x^2)]}}=-\int_{\frac{1}{2}\pi}^{a} \frac{\sin\phi\, d\phi}{\sqrt{[(1-\cos^2\phi)(1+\cos^2\phi)]}}$$

$$=\int_a^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{[2-\sin^2\phi]}}=\frac{1}{\sqrt{2}}\int_a^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{[1-\frac{1}{2}\sin^2\phi]}}$$

$$=\frac{1}{\sqrt{2}}\left[F\left(\frac{\pi}{2}\right)-F(a)\right] \text{ modulus } \frac{1}{\sqrt{2}}$$

where $F\left(\frac{\pi}{2}\right)$ and $F(a)$ are Legendre's elliptic integrals of the first kind.

As a numerical example take $\theta=\frac{1}{2}$. $\therefore a=\frac{1}{3}\pi$.

$A_\theta=\frac{1}{\sqrt{2}}[1.8451-1.1424]=0.5032$, using the four-place "funktionentafeln" of Jahnke and Emde. As a verification, expand $\frac{1}{\sqrt{(1-x^4)}}$ by the binomial theorem and integrate:

$$A_\theta=\int_0^\theta \frac{dx}{\sqrt{[1-x^4]}}=\int_0^\theta(1+\tfrac{1}{2}x^4+\tfrac{3}{8}x^8+\tfrac{5}{16}x^{12}\ldots)dx=\theta+\frac{\theta^5}{10}+\frac{\theta^9}{24}+\tfrac{5}{208}\theta^{13}.$$

On substituting $\theta=\frac{1}{2}$ we get .5032 for A_θ as before.

II. Solution by the PROPOSER.

Let $x=\tan\omega$. Then $dx=\frac{d\omega}{\cos^2\omega}$. Substituting these values in the integral expression, we have

$$A_\theta=\int_0^\omega \frac{d\omega}{\cos^2\omega\sqrt{[(1-\tan^2\omega)(1+\tan^2\omega)]}}$$

$$=\int_0^\omega \frac{d\omega}{\sqrt{[\cos^2\omega-\sin^2\omega]}}=\int_0^\omega \frac{d\omega}{\sqrt{[1-2\sin^2\omega]}},$$

where $\tan\omega=\theta$ is used for the upper limit.

Now let $\sqrt{2}\sin\omega=\sin\phi$; then $1-2\sin^2\omega=\cos^2\phi$ and

$$d\omega = \frac{\cos\phi\, d\phi}{\sqrt{2}\sqrt{[1-\tfrac{1}{2}\sin^2\phi)]}}.$$

Whence $A_\theta = \tfrac{1}{2}\sqrt{2}\int_0^\phi \frac{d\omega}{\sqrt{[1-\tfrac{1}{2}\sin^2\phi]}} = \tfrac{1}{2}\sqrt{[2]}F(\tfrac{1}{2}\sqrt{2},\ \phi)$.

The amplitude ϕ is determined from $\theta=\tan\omega$, $\cos^2\omega=\frac{1}{1+\theta^2}$, $\sin^2\omega=\frac{\theta^2}{1+\theta^2}$, $\sin^2\phi=\frac{2\theta^2}{1+\theta^2}$, $\cos^2\phi=\frac{1-\theta^2}{1+\theta^2}$. Hence, $\tan^2\phi=\frac{2\theta^2}{1-\theta^2}$.

Referring to problem 303,

$$A=\int_0^1 \frac{dx}{\sqrt{[1-x^4]}} = \frac{\sqrt{\pi}}{4}\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})},$$

the well known formula $\Gamma(\theta)\Gamma(1-\theta)=\pi/\sin\pi\theta$, gives in our case $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})=\pi\sqrt{2}$. Whence, $\Gamma(\frac{3}{4})=\frac{\pi\sqrt{2}}{\Gamma(\frac{1}{4})}$, and therefore, $A=\frac{[\Gamma(\frac{1}{4})]^2}{4\sqrt{[2\pi]}}$.

This combined with the result in above, $A=\tfrac{1}{2}\sqrt{[2]}F^I(\tfrac{1}{2}\sqrt{2})$ yields $\Gamma(\frac{1}{4})=2\sqrt[4]{[\pi]}\{F^I(\tfrac{1}{2}\sqrt{2})\}^{\frac{1}{4}}=3.62561$, *0.5593811.

And similarly, $\Gamma(\frac{3}{4})=\tfrac{1}{2}\sqrt{[2]}\sqrt[4]{[\pi^3]}\{F^I(\tfrac{1}{2}\sqrt{2})\}^{-\frac{1}{4}}=1.225416$, *0.0882838.

These expressions for $\Gamma(\frac{1}{4})$ and $\Gamma(\frac{3}{4})$ in Legendre's F-functions are to my mind by far the most important consequences of evaluating integral A in gamma-functions. Without this evaluation $\Gamma(\frac{1}{4})$ and $\Gamma(\frac{3}{4})$ can be determined only by computing their natural logarithms by inconvenient series.

Also solved by V. M. Spunar, C. N. Schmall, and J. Scheffer.

MECHANICS.

253. Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Stroud, England.

R_1 and R_2 are ranges on a horizontal plane of particles projected with given velocity from A on the plane to pass through B. Show that $a(R_1+R_2)-R_1R_2=\frac{a^4}{c^2}$, where $c=AB$ and a is the horizontal projection of AB.

III. Solution by the PROPOSER.

If α, α_1 be angles of projection and β the angle AB makes with the horizontal, and v the velocity of projection, then $\alpha_1=\frac{1}{2}\pi-(\alpha-\beta)$.

And $\cos\beta=a/c$; $R_1.g=a^2\sin2\alpha$; $R_2.g=a^2\sin2(\alpha-\beta)$.

$$AB=c=\frac{2v^2}{g}\frac{\cos\alpha\sin(\alpha-\beta)}{\cos\beta}=\frac{2v^2c^2}{ga^2}\sin(\alpha-\beta)\cos\alpha.$$

$$\therefore a(R_1+R_2)=\frac{av^2}{g}[\sin 2\alpha+\sin 2(\alpha-\beta)]=\frac{2av^2}{g}\sin 2(\alpha-\beta)\cos\beta$$

$$=\frac{2av^2}{g}[\sin\alpha\cos(\alpha-\beta)+\cos\alpha\sin(\alpha-\beta)]\cos\beta$$

$$=\frac{a^4}{c^2}\left(\frac{\sin\alpha\cos(\alpha-\beta)+\cos\alpha\sin(\alpha-\beta)}{\cos\alpha\sin(\alpha-\beta)}\right).$$

$$R_1R_2=\frac{v^4}{g^2}\sin 2\alpha\sin 2(\alpha-\beta)=\frac{4v^4}{g^2}\sin\alpha\cos\alpha\sin(\alpha-\beta)\cos(\alpha-\beta)$$

$$=\frac{a^4}{c^2}\frac{\sin\alpha\cos(\alpha-\beta)}{\cos\alpha\sin(\alpha-\beta)}.$$

$$\therefore a(R_1+R_2)-R_1R_2=\frac{a^4}{c^2}\frac{\cos\alpha\sin(\alpha-\beta)}{\cos\alpha\sin(\alpha-\beta)}=\frac{a^4}{c^2}.$$

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

Edited by Dr. G. E. Wahlin, University of Illinois.

183. Proposed by MERTON T. GOODRICH, Dixfield, Maine.

What relations must exist between the quantities A, B, and C in the harmonic ratio $\dfrac{AB}{(A+B+C)(-C)}=-1$ so that they will be positive integers.

II. Solution by the PROPOSER.

Solving the given equation for A, we have $A=\dfrac{(B+C)C}{B-C}$. Since A, B, and C are positive, we see that $B-C$ must be positive. Since A is an integer, either $\dfrac{C}{B-C}=\dfrac{M}{N}=K$, or $\dfrac{B+C}{B-C}=\dfrac{M'}{N'}=K'$, where M and N, M' and N' are integers, and M prime to N, and M' prime to N'. From the first of these equations, $C=K(B-C)$. Since C and $B-C$ are positive, K must be positive. Solving this last equation for B, we have $B=\dfrac{(K+1)C}{K}=\dfrac{(M+N)C}{M}$. Since M and N are relatively prime, $M+N$ is prime to M. Hence, B being an integer, C is divisible by M. That is, $C=MD'=MND'/N=KD$, where $D=ND'$, and D' and hence D are positive integers.

Substituting KD for C in the expression for B, $B=(K+1)D=C+D$. Substituting KD for C and $(K+1)D$ for B in the expression for A, $A=(2K+1)KD=(2K+1)C$. Putting $\dfrac{M}{N}$ for K, $A=\dfrac{(2M+N)MD}{N^2}$. This tells us that if N is odd $D=N^2.\lambda$; but if N is even, $2M+N$ is even and then $D=\dfrac{N^2.\lambda}{2}$. Hence we have this set of relations: $A=(2K+1)C$, $B=C+D$,

$C=KD$, where K is a positive fraction in its lowest terms or a positive integer, and D is a positive integer as above determined.

The second condition is not independent of the first, because, if K' is substituted for $2K+1$ and $2D'$ for D, in the above set of relations, then we have

$$\frac{B+C}{B-C}=\frac{D'K'+D+'D'K'-D'}{D'K'+D'-D'K'+D'}=\frac{2D'K'}{2D'}=K',$$

which is the second condition. Hence the set of values found above includes all the possible relations which make A, B, and C positive integers. The values of A and B may be interchanged, and A, B, and C may each be multiplied by a common factor without changing the value of the original ratio.

Also solved by A. H. Holmes.

184. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Prove that $\frac{\pi}{12}=\tan^{-1}\frac{1}{2^2}+\tan^{-1}\frac{1}{8^2}+\tan^{-1}\frac{1}{30^2}+\tan^{-1}\frac{1}{112^2}+\ldots$, where 2, 8, 30, 112..., is a recurring series with the recursion formula $u_n=4u_{n-1}-u_{n-2}$.

Solution by the PROPOSER.

If we reduce $\sqrt{3}$ to a continued fraction, we get for the convergents,

$$\frac{1}{1},\ \frac{2}{1},\ \frac{5}{3},\ \frac{7}{4},\ \frac{19}{11},\ \frac{26}{15},\ \frac{71}{41},\ \frac{97}{56},\ \frac{265}{153},\ \frac{362}{209},\ \cdots$$

The alternate convergents,

$$\frac{1}{1},\ \frac{5}{3},\ \frac{19}{11},\ \frac{71}{41},\ \frac{265}{153},\ \cdots$$

are formed by taking the ratios of the corresponding terms of the two recurring series

$$1,\ 5,\ 19,\ 71,\ 265,\ \ldots$$
$$1,\ 3,\ 11,\ 41,\ 153,\ \ldots$$

both having the same scale of relation $u_n=4u_{n-1}-u_{n-2}$.

We find by the usual methods that the nth terms of the two series are

$$\frac{\alpha^n-\beta^n}{\alpha-\beta}+\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}=u_n+u_{n-1},\ \text{and}\ \frac{\alpha^n-\beta^n}{\alpha-\beta}-\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}=u_n-u_{n-1},$$

where α and β are the roots of the equation $x^2-4x+1=0$.

Taking the differences of the $\tan^{-1}$ function of the reciprocals of the convergents, we have

$$\tan^{-1}\frac{1}{1}-\tan^{-1}\frac{3}{5}=\tan^{-1}\frac{1}{4}$$

$$\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{11}{19}=\tan^{-1}\frac{1}{64}$$

$$\tan^{-1}\frac{11}{19}-\tan^{-1}\frac{41}{71}=\tan^{-1}\frac{1}{900}$$

$$\tan^{-1}\frac{41}{71}-\tan^{-1}\frac{153}{265}=\tan^{-1}\frac{1}{12544}$$

. . . . . .

$$\tan^{-1}\frac{u_n-u_{n-1}}{u_n+u_{n-1}}-\tan^{-1}\frac{u_{n+1}-u_n}{u_{n+1}+u_n}=\tan^{-1}\frac{1}{4u_n^2}$$

. . . . . .

Adding, we have

$$\tan^{-1}\frac{1}{1}-\tan^{-1}\frac{1}{\sqrt{3}}=\frac{\pi}{4}-\frac{\pi}{6}=\frac{\pi}{12}=\tan^{-1}\frac{1}{2^2}+\tan^{-1}\frac{1}{8^2}+\tan^{-1}\frac{1}{30^2}+\ldots$$

$$+\tan^{-1}\frac{1}{(2u_n)^2}+\ldots$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

360. Proposed by CHARLES C. GROVE, Columbia University, New York.

A bridge club of 28 members has 27 meetings. There are 7 tables with 4 members at each table. Can the players be so arranged that at the end of the season (27 meetings) each member will have played *with* every other member one game and *against* every other member two games, one game meaning one meeting; and how?

361. Proposed by C. E. GITHENS, Ph. D., Wheeling, W. Va.

Find three integral values for $[-10+9\sqrt{(-3)}]^{\frac{1}{3}}+[-10-9\sqrt{(-3)}]^{\frac{1}{3}}$. A solution not involving a cubic is desired.

362. Proposed by JAMES F. LAWRENCE, Stillwater, Okla.

Show that the number of solutions in positive integers, zero included, of the equation $x+2y+3z=6n$, is $3n^2+3n+1$.

GEOMETRY.

393. Proposed by S. LEFSCHETZ, Clarke University.

Draw a triangle having a given angle, and with its vertices on three given concentric circles.

394. Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Stroud, England.

The joins of the excentres to the corresponding vertices of the pedal triangle are concurrent.

395. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

From a point P without a rectangular field ABC the distances PA, PB, PC measured to the corners are respectively 70, 40, 60 chains. What is the area of the field?

CALCULUS.

316. Proposed by C. N. SCHMALL, New York City.

$$\int_0^\infty \frac{\cos ax}{1+x^2}dx=\tfrac{1}{2}\pi e^{-a}=\int_0^\infty \frac{x\sin ax}{1+x^2}dx.$$

(From Bromwich, *Theory of Infinite Series*, p. 442, ex. 5, and also from Carslaw, *Fourier's Series*, p. 113, ex. 12.) Prove this by any method.

317. Proposed by C. N. SCHMALL, New York City.

A generating line of a right circular cylinder passes through the center of a sphere. The diameter of the cylinder is less than the radius of the sphere. Show that the surface of the cylinder included within the sphere is given by an elliptic integral.

MECHANICS.

263. Proposed by C. N. SCHMALL, New York City.

A railroad car is rounding a curve of radius r with a velocity v, $2d$ being the distance between the rails. If h be the height of its center of gravity above the rails, and g have its usual meaning, show that the weight of the car is divided between the outer and inner rails in the ratio $\frac{dgr+v^2h}{dgr-v^2h}$.

264. Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Stroud, England.

Three particles, weights ω_1, ω_1, ω_2, and three light strings of equal length connecting them, lie in a vertical smooth circular tube. Discuss the possible positions of stable and unstable equilibrium. If a slight displacement takes place in the unstable position, find the maximum ensuing velocity.

NOTES AND NEWS.

The new Spanish Mathematical Society, of Madrid, Spain, has issued the third number of its journal called *Revista de la Sociedad Matematica Espanola.* According to a statement on page 96 of this journal the membership of this society has grown very rapidly, having surpassed 370 on the date of the first reunion which was held in Madrid on the 28 day of last June. M.

B. G. Teubner, of Leipzig, Germany, has published the first volume of *Euler's Complete Works.* It has been estimated that these works would comprise about forty large volumes and that the cost of publication would be about eighty thousand dollars. It may be remembered that the people of Switzerland and various learned societies, including the American Mathematical Society, contributed liberally towards this publication. Euler has been the most prolific mathematical writer up to the present time, and his works have had a very important influence on the development of mathematics during almost two centuries. This first volume is in German and deals with elementary arithmetic and elementary algebra. M.

A recent number of the *Rendiconti del Circolo Matematico di Palermo* contains the following interesting statistics in reference to the principal mathematical societies existing in 1911. The largest of these is the Circolo Matematico di Palermo, founded in 1884, and having a membership of 810 in August of the present year. Only a little more than one-third of these members, 286, resided in Italy. The second society in order of membership is the Deutsche Mathematiker-Vereinigung, founded in 1890, and having a membership of 751 at the beginning of the present year. Considerably more than one-half of these, 449, resided in the empire of Germany. The American Mathematical Society, founded in 1888, and having a membership of 641 at the beginning of the present year, is the third in order of size. As only 53 of these members resided outside of the United States, this society has a considerably larger resident membership than any other. The oldest society is the Mathematische Gesellschaft in Hamburg, founded in 1690, and having a membership of 111 at the beginning of the present year. M.

BOOKS.

Monographs on Topics of Modern Mathematics Relevant to the Elementary Field. Edited by J. W. A. Young. 8vo. viii+416 pages. Price, $3.00. New York: Longmans, Green & Co.

Titles and Authors.—I. The Foundation of Geometry. By Oswald Veblen, Ph. D., Professor of Mathematics in Princeton University. II. Modern Pure Geometry. By Thomas F. Holgate, Ph. D., LL. D., Professor of Mathematics in Northwestern Univer-

sity. III. Non-Euclidean Geometry. By Frederick S. Woods, Ph. D., Professor of Mathematics in the Massachusetts Institute of Technology. IV. The Fundamental Propositions of Algebra. By Edward V. Huntington, Ph. D., Assistant Professor of Mathematics in Harvard University. V. The Algebraic Equation. By G. A. Miller, Ph. D., Professor of Mathematics in the University of Illinois. VI. The Function Concept and the Fundamental Notions of the Calculus. By Gilbert Ames Bliss, Ph. D., Associate Professor of Mathematics in the University of Chicago. VII. The Theory of Numbers. By J. W. A. Young, Ph. D., Associate Professor of the Pedagogy of Mathematics in the University of Chicago. VIII. Constructions with Ruler and Compasses; Regular Polygnons. By L. E. Dickson, Ph. D., Professor of Mathematics in the University of Chicago. IX. The History and Transcendence of π. By David Eugene Smith, Ph. D., LL. D., Professor of Mathematics in Teachers College, Columbia University.

Editor's Preface. "The purpose of this collection of monographs may be indicated by the following citation from the letter that was sent to those who were requested to act as authors.

"Among the varions publications on mathematics that are being made, it would seem that there is room for a serious effort to bring within reach of secondary teachers (in service or in training), college students, and others at a like stage of mathematical advancement, a scientific treatment of some of the regions of advanced mathematics that have points of contact with the elementary field. Undoubtedly one of the most crying needs of our secondary instruction in mathematics to-day, is that the scientific attainments of the teachers be enlarged and their mathematical horizon widened; and I believe that there is a large body of earnest teachers and students that are eager to extend their mathematical knowledge if the path can be made plain and feasible for them.

"A volume of monographs dealing with selected topics of higher mathematics might well be a useful contribution to the meeting of this need. Such monographs would aim to bring the reader into touch with some characteristic results and viewpoints of the topics considered, and to point out their bearings on elementary mathematics. They would therefore contain:

(1) A considerable body of results proved in full, so that the reader can materially extend his mathematical acquisitions by the reading of the monograph alone.

(2) Statement without proof of some leading methods and results, so as to give a bird's-eye view of the subject.

(3) A small number of references indicating what the reader may profitably take up after he has mastered the contents of the monograph."

Both the plan itself, and the invitation to act as author, were most cordially received; work on the monographs was promptly begun, has been carried through substantially as planned, and the results are now presented in this book."

Vocational Algebra. By George Wentworth and David Eugene Smith. 12mo. Cloth. 88 pages, Illustrated. Price, 50 cents. New York and Chicago: Ginn & Co.

"The time has arrived when algebraic language has such a well-defined place in trade journals, artisans' manuals, and handbooks of business that the workman in the shop and the business man in the office have each a practical need to interpret it. The growth of industrial and commercial classes, one of the most significant features of our present work in education, helps to create such a need, and the shop itself is not far behind the school in making it known. It is to meet the demand for the essentials of algebra thus required in preparation for the shop and commerce that 'Vocational Algebra' has been written. It presents exactly the topics that are needed in vocational classes, no more and no less. Any one who has mastered it will be able to understand all the algebra of ordinary trade or business. The book contains a wide range of general vocation problems, but is free from mere puzzles and from those technicalities with which neither teacher nor stu-

dent should be expected to be familiar. In addition to this applied work it offers a sufficient amount of drill in abstract algebraic forms to insure a mastery of all the principles involved.

"Throughout the formula and equation are continually reviewed and constitute the leading feature of every chapter."

The Elements of Plane and Spherical Trigonometry. By John Gale Hun and Charles Ranald MacInnes. 8vo. Cloth, 100 pages of Text, 92 pages of Tables, and 13 pages of Explanation of Tables. Price, $1.35. New York: The Macmillan Co.

The material in this book has been used in pamphlet form from three to four years in Princeton University. The authors have attempted to present the essentials of subject in as brief and clear a manner as possible. Some space is devoted to the drawing of graphs of simple equations in polar coördinates because it is thought that such problems aid the student in getting a clearer idea of the way in which the functions vary with the change of angle. Many of the proofs are very brief though easily understood by the average student. F.

The Hindu-Arabic Numerals. By David Eugene Smith, Professor of Mathematics, Teachers College, Columbia University, and Louis C. Karpinski, Instructor in Mathematics, University of Michigan, Ann Arbor. 8vo. Cloth, 160 pages. Price, $1.25. Boston: Ginn & Co.

"Although it has long been known that the numerals ordinarily employed in business and commonly attributed to the Arabs, are not of Arabic origin, and although numerous monographs have been written concerning their derivation, no single work has yet appeared in which the complete story of their rise and development has been told. In the preparation of this treatise the authors have examined every important book and monograph that has appeared upon the subject, consulting the principal libraries of Europe as well as America, examining many manuscripts, and sifting the evidence with greatest care. The result is a scholarly discussion of the entire question of the origin of the numerals, the introduction of the zero, the influence of the Arabs, and the spread of the system about the shores of the Mediterranean and into the various countries of Europe.

The work is illustrated with numerous facsimilies from early inscriptions and manuscripts, most of which have not heretofore been published in connection with this subject, and all of which contribute to a very marked degree to an understanding of the problem.

Such a contribution to history, to mathematics, and to education bearing the names of two authors of such prominence in the history of mathematics, should find a place in every library of importance, and upon the shelves of all who are interested in education in its broadest aspect."

The Progress of Physics, During the 33 years (1875-1908). Four Lectures Delivered to the University of Calcutta During March, 1908. By Arthur Schuster, F. R. S., Ph. D., (Heidelberg), Sc. D. (Cantab), D. Sc., (Manchester and Calcutta), Dès. Sc. (Geneva).

8vo. Cloth, x+104 pages. Price $1.25. Cambridge, England: The University Press: G. P. Putnam's Sons, American Agents.

When one has once begun the reading of these four lectures, one will not want to lay the book aside until its perusal is completed. The author's personal contact and acquaintance with the greatest scientist of the past thirty years, makes his descriptions of their experiments and discoveries of more than ordinary value. The frontispiece is an excellent portrait of James Clerk-Maxwell. F.

A Treatise on Electric Theory and the Problem of the Universe. Considered from the Physical Point of View, with Mathematical Appendices. By G. W. de Tunzelmann, B Sc., Member of the Institution of Electrical Engineers, Formerly Professor of Physics and Astronomy, H. M. S. "Britannia," Dartmouth. 8vo. Cloth. xxxi+654 pages. Price, $4.50. Philadelphia: J. B. Lippincott & Co.

This volume is profoundly interesting and instructive. It deals with the most absorbing questions in the whole realm of matter. Some idea of the range of subjects discussed may be learned from the subjects of the twent-four chapters into which the book is divided. Chapter I, Fundamental Electrical Phenomena; chapter II, Units and Measurements; chapter III, Meaning and Possibility of a Mechanical Theory of Electricity; chapter IV, The Ether; chapter V, The Ether as a Framework for Absolute Motion; chapter VI, Relations Between Ether and Moving Matter; chapter VII, Conditions in Gases and Dielectrics; chapter VIII, The Faraday-Maxwell Theory; chapter IX, The Electron Theory; chapter X, Magnetism and the Dissipation of Energy; chapter XI, Contact Electrification and Electrolysis; chapter XII, Optical Phenomena; chapter XIII, The Mechanism of Radiation; chapter XIV, Metallic Conduction and Radiation; chapter XV, General Phenomena of Radioactivity; chapter XVI, The Three Principal Types of Radioactivity; chapter XVII, Transformation of Radioactive Substances; chapter XVIII, The Ages of the Earth and Sun, and the Probable Origin of Radio-active Substances; chapter XIX, The Solar Corona, the Aurora and Comets' Tails; chapter XX, Radio-activity in Stars and Nebulae; chapter XXI, Arrangments and number of Electrons in an Atom; chapter XXII, Change in the Aspect of Fundamental Mechanical Principles; chapter XXIII, Gravitation and Cohesion; chapter XXIV, The Place of Mind in the Universe. While one may not accept all the theories advanced in this volume, one cannot help being interested in them, for further investigation and study may raise them to the rank of definite laws and principles. F.

Optical Geometry of Motion. A New View of the Theory of Relativity. Alfred A. Robb, M. A., Ph. D. Pamphlet, 32 pages. Cambridge: W. Heffer & Sons.

An essay which puts the theory of Relativity in mathematical dress. F.

Elements of Trigonometry. By Daniel A. Murray, Ph. D., Professor of Applied Mathematics in McGill University. 8vo. Cloth. ix+136 pages. Price, 75 cents. New York: Longmans, Green and Co.

This is a shorter course than the author's former text-book entitled *Plane Trigonometry.* The omission of many notes and of several topics in the former and a more condensed treatment of others enables him to present the essentials in this abbreviated form. F.

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THE TENTH PERFECT NUMBER.

By R. E. POWERS, Denver, Colorado.

A number which is equal to the sum of all its divisors is called a "perfect number." Thus, the divisors of 6 are 1, 2, and 3, the sum of which is equal to 6; the divisors of 28 are 1, 2, 4, 7, and 14, whose sum is 28. Euclid (IX, 36) proved that if 2^p-1 is prime, then $2^{p-1}(2^p-1)$ is a perfect number, and no other perfect numbers are known. In 1644 Mersenne, in the preface to his *Cogitata Physico-Mathematica*, stated, in effect, that the only values of p not greater than 257 which make 2^p-1 prime are 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, and 257. Regarding these "Mersenne's Numbers" (2^p-1), W. W. Rouse Ball, in his *Mathematical Recreations and Essays* (4th Edition, pages 262, 263, 269), says:

"I assume that the number 67 is a misprint for 61. With this correction, we have no reason to doubt the truth of the statement, but it has not been definitely established. . . . Seelhoff showed that 2^p-1 is prime when $p=61$, . . . and Cole gave the factors when $p=67$. . . . One of the unsolved riddles of higher arithmetic . . . is the discovery of the method by which Mersenne or his contemporaries determined values of p which make a number of the form 2^p-1 a prime. . . . The riddle is still, after nearly 250 years, unsolved."

No exception to Mersenne's assertion (corrected by the substitution of 61 for 67) is known at the present time. Below we show, however, that $2^{89}-1$ is a prime number, contrary to his statement.

Following is a list of Mersenne's Numbers thus far proved to be prime, with the corresponding perfect numbers: